Direct and Inverse Problem in Supersonic Axisymmetric Flow

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Introduction

SUPERSONIC inviscid flow can generally be solved by the method of characteristics or by shock-capturing methods.[1-5] The method of characteristics computes the flow along characteristics and uses the Rankine-Hugoniot conditions at the shock. This method has the advantage of accuracy, but is regarded as complex and computationally inefficient, especially in regions of near coalescence of the two sets of characteristics.[1,3] In shock-capturing methods the shock is smeared over several grid points, where oscillations can occur and the scheme loses accuracy. However, due to their directness and computational ease, shock-capturing methods have been preferred in recent years.

In this Note, we develop an efficient method using the characteristics and streamlines of a flow. These are used as coordinates and the flow quantities are expressed in terms of Riemann functions. A scheme is obtained which is significantly more efficient and accurate than shock-capturing methods for flow over axisymmetric bodies. Since streamlines form one of the coordinates, we naturally obtain a body-fit system. It is also a truly shock-fit coordinate system. Not only are the Rankine-Hugoniot conditions used, but the shock lies exactly on grid points also.

The success of the present method in the two-dimensional case rests on the discovery of an accurate and simple approximation.[6] In the axisymmetric case, a similarly accurate and simple approximation has eluded us. The approximation presented herein is simple, but generally not as accurate as that for the two-dimensional case. A better iterative procedure has been developed to compensate for this weakness. As was the case for the two-dimensional flow,[7] our method is well suited for the inverse design problem (i.e., given $M_0>1$ and the pressure distribution on the body, find the shape of the body and the flow everywhere).

Formulation

We consider axisymmetric flow with incident Mach number $M_0>1$ and shock attached at the tip. The characteristic equations can be written in terms of entropy s, flow angle θ, Mach angle $\mu=\sin^{-1}(1/M)$ (M=local Mach number), pressure p, density ρ, and velocity q as follows[8]:

$$ds=0 \text{ on } \frac{dr}{dx}=\tan\theta \tag{1}$$

$$d\theta \pm \frac{dp}{\rho q^2 \tan\mu} = \mp \frac{\sin\theta\sin\mu}{\sin(\theta\pm\mu)}\frac{dr}{r} \text{ on } C^{\pm}: \frac{dr}{dx}=\tan(\theta\pm\mu) \tag{2}$$

At the body $r=f(x)$ we impose the boundary condition $\tan\theta=f'(x)$. Jumps across a shock are given by the Rankine-Hugoniot conditions.[9] If we denote the shock angle by η, the position of the shock is governed by

$$\frac{dr}{dx}=\tan\eta \tag{3}$$

We introduce new coordinates (α,β) through

$$r_\beta = x_\beta \tan\theta \tag{4}$$

$$r_\alpha = x_\alpha \tan(\theta+\mu) \tag{5}$$

so that constant α refers to streamlines, and constant β refers to C^+ characteristics. Expressed in these coordinates, Eq. (1) is simply $s_\beta=0$, while Eq. (2) becomes

$$\frac{\partial}{\partial\alpha}R^+ = (\theta+P_{(\mu)})_\alpha = \frac{\sin 2\mu}{2\gamma}s'(\alpha) - \frac{\tan\theta\tan\mu}{\tan\theta+\tan\mu}\frac{r_\alpha}{r} \tag{6}$$

$$DR^- = D(\theta-P_{(\mu)}) = -\frac{\sin 2\mu}{2\gamma}s'(\alpha) + \frac{\tan\theta\tan\mu}{\tan\theta-\tan\mu}\frac{D_r}{r} \tag{7}$$

where $P(\mu)=(\lambda)^{1/2}\tan^{-1}[(\lambda)^{1/2}\tan\mu]-\mu$ is the Prandtl angle, $\lambda=(\gamma+1)/(\gamma-1)$, and

$$D=\frac{\partial}{\partial\alpha} - \frac{2\tan\theta}{\tan\theta+\tan\mu}\frac{r_\alpha}{r_\beta}\frac{\partial}{\partial\beta}$$

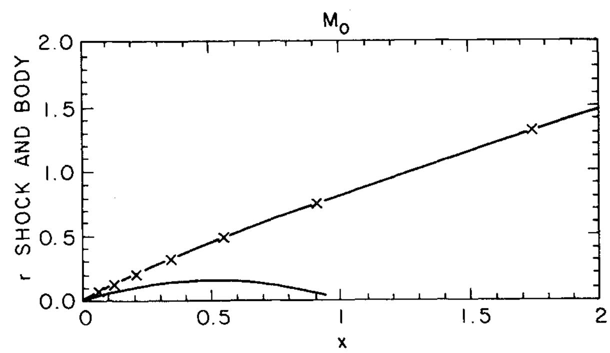

Fig. 1 Body and shock of 30%-thick parabolic body at $M_0=2$.

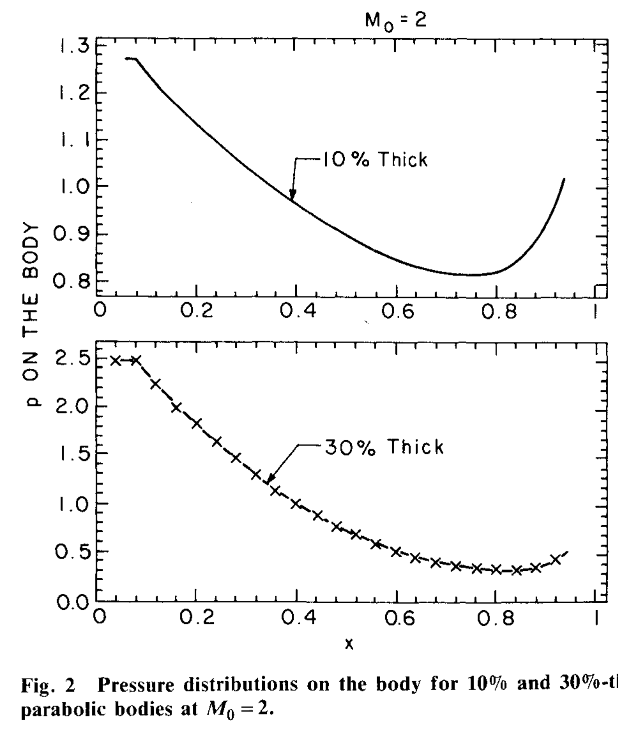

Fig. 2 Pressure distributions on the body for 10% and 30%-thick parabolic bodies at $M_0=2$.

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Here $\gamma = 1.4$. An alternate form of Eq. (7) used later is as follows:

$$R_\beta^- = (1 - \tan\theta\tan\mu)\frac{x_\beta}{x_\alpha}\theta_\alpha + \tan\mu\frac{r_\beta}{r} \tag{8}$$

It is important to note that the dynamical equations must be augmented by the coordinate equations (4) and (5). This transformation is still undetermined up to two arbitrary functions. We fix one of these functions by taking $x(0,\beta) = \beta$ at the body $\alpha = 0$. It therefore follows that

$$r(0,\beta) = f(\beta), \quad \theta(0,\beta) = \tan^{-1} f'(\beta) \tag{9}$$

To determine the second arbitrary function we fix the shock by

$$\frac{d\alpha}{d\beta} = 1 \tag{10}$$

The above formulation must be modified slightly in order to treat the inverse design problem. For this problem, pressure is specified on an unknown body. To accommodate this boundary condition, we obtain from Bernoulli's equation and the perfect gas law

$$\mu = \sin^{-1}\left[\left(\frac{\gamma-1}{2} \times \frac{\exp\{[(\gamma-1)/\gamma](s + \ell n p)\}}{1 + [(\gamma-1)/2]M_0^2 - \exp\{[(\gamma-1)/\gamma](s + \ell n p)\}}\right)^{1/2}\right] \tag{11}$$

Since the entropy s is constant along streamlines and is known behind the tip shock, the right-hand side of Eq. (11) is determined. Therefore, instead of Eq. (9), we have Eqs. (4) and (11) on the body. The flow above the body can be calculated exactly as before. Hence the inverse problem becomes a direct problem by the method presented here.

Numerical Procedure

Near the tip of the body, the flow is taken to be flow over a cone. For the rest of the flow, we use a marching scheme along each column of β. A good approximation is first obtained at the points on that column. The governing equations are then iterated until an error tolerance is satisfied before we proceed to the next column.

It is worth noting that we are free to choose any mesh size of β, even when the C^+ and C^- characteristics are nearly parallel. Due to Eq. (10), we are also not forced to take very small mesh sizes in α either.

Approximation Solution

Eggers and Savin[10] noted that hypersonic flow over asymmetric bodies can be approximated as locally two-dimensional and, hence, shock-expansion theory (R^- constant along an entire steamline) is valid. Indeed, for hypersonic flow, along C^+ we have $s'(\alpha)$ large and r_α small so that Eqs. (7) and (8) are well approximated by two-dimensional theory. At lower Mach numbers, this approximation is no longer valid. However, in our marching scheme, we need only to approximate one grid point away, and to assume that R^- changes slowly along a streamline. Combining Eqs. (7) and (9),

$$R_\beta^- = x_\beta(1 - \tan\theta\tan\mu)\frac{\sin 2\mu}{2\gamma}\frac{Q_\alpha}{x_\alpha}$$

$$Q = s + \frac{\gamma}{\gamma-1}\ell n\left(1 + \frac{\gamma-1}{2\sin^2\mu}\right)$$

Here $Q_\alpha = 0$ is an approximation consistent with the approximation $R_\beta = 0$. These are the two ordinary differential equations which give us the initial guesses in the iteration scheme.

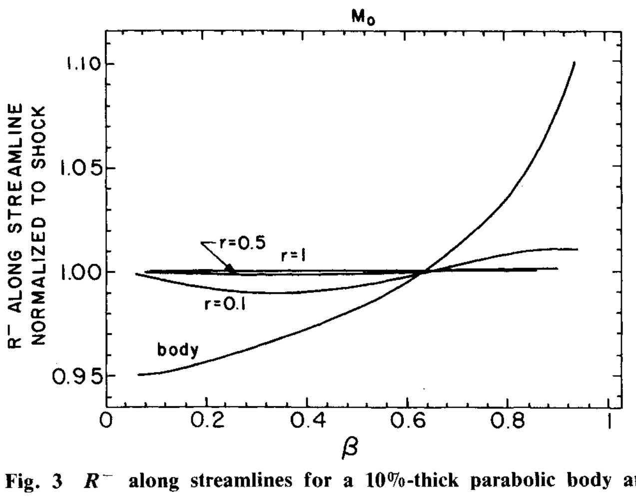

Fig. 3 R^- along streamlines for a 10%-thick parabolic body at $M_0 = 2$.

Results and Discussions

Calculations of flow over bodies of various shapes for a range of Mach numbers have been performed. The average number of iterations (generally between 2 and 4 for an error tolerance of 10^{-4}) used per point decreases as the total number of points used increases. In view of our special coordinate system, few points are required to describe the entire flowfield. Grid points are spaced appropriately according to the natural variations of the flow. This is demonstrated in Fig. 1 with a 30%-thick parabolic body at $M_0 = 2$, where we compare a calculation using 248 points in the whole flowfield to one using 829 points.

Figure 2 shows the pressure distributions on two parabolic bodies. The flow near the tip is conical, hence pressure is constant there. The presence of a pressure minimum on the body should be noted. No such minimum appears in two-dimensional parabolic wings. Consistent with this is the fact that the minimum moves further down along the body as M_0 or the thickness increases, because thicker bodies are more two-dimensional. This should be of some interest since the presence of a pressure minimum in some cases can be an indication of flow separation.

Figure 3 shows the changes of R^- (normalized to its value at the shock) along streamline for a 10% body at $M_0 = 2$. The streamlines shown lie on the body and at about 0.1, 0.5, and 1 body length away from the axis. As one can see, R^- changes monotonically along the body. The approximation R^- constant along the whole body is poor, but R^- does become nearly constant at less than a body length away.

To test the method for solving the inverse problem, a direct problem was solved, and the resulting pressure distribution on the body was used in the inverse method. The agreement was excellent and the computational time is comparable to the direct problem.

Conclusion

A streamlines-characteristics coordinate system for axisymmetric flow has been presented. In our coordinate system the body lies along one coordinate, and the shock is a straight line with grid points falling exactly on it. Rankine-Hugoniot conditions are used at the shock. Consequently, the scheme is inherently accurate. Relatively few points are required to describe the entire flowfield since grid points are spaced according to the natural variation of the flow. Even in regions of near coalescence of the C^+ and C^- characteristics, we are not forced to take very small mesh sizes. Starting from a simple initial guess, the scheme converges rapidly to the exact solution. Because so few points and iterations are needed, the method is computationally very efficient. Our body-fit coordinate system also allows us to solve the inverse design problem with ease, and due to our coordinate system the inverse problem becomes a direct problem. Our results for thin bodies at low Mach numbers show there is a pressure minimum on the bodies, which can imply flow separation in some cases.

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Forced Convection over Rotating Bodies with Blowing and Suction

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Introduction

SPINNING an axisymmetric body in a forced flow field in order to develop rotating systems for enhancing the heat-transfer rate is important in the analysis of problems involving projectile motion and rotary machine design. Several studies of this problem[1,2] have yielded very effective solutions for moderate or high Prandtl numbers and for small values of the rotation parameter. In order to avoid the difficulties encountered in previous methods, Lee et al.[3] have analyzed the momentum and heat-transfer rates through laminar boundary layers over rotating isothermal bodies by employing Merk's series expansion technique.

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The present study is an extension of Ref. 3 and deals with an isothermal (ISO) or constant-heat-flux (CHF) rotating body with blowing and suction. An efficient finite difference method is employed. These results are compared with those of Lee et al.[3] and Hoskin.[2]

Analysis

This paper is an extension of Ref. 3 to include injection and suction at the surface. The basic governing equations are identical to those of Ref. 3 with the following boundary conditions:

$$u=0, \quad v=v_w, \quad \omega=r\Omega, \quad T=T_w$$

$$\text{or} \quad k\frac{\partial T}{\partial y}=-q_w, \qquad \text{for } y=0$$

$$u=Ue, \quad v=\omega=0, \quad T=T_\infty, \quad \text{for } y\to\infty \tag{1}$$

To solve the basic boundary-layer equations, we introduce pseudosimilarity variables (ξ,η), dimensionless rotating velocity function g, and dimensionless temperature θ (ISO), which are the same as those in Ref. 3. Besides, we also define a stream function $\psi(x,y)$ (for permeable surfaces) and a dimensionless temperature θ (CHF) as follows:

$$\theta=(T-T_\infty)Re_L^{1/2}/(q_wL/k) \tag{2}$$

$$\psi(x,y)=u_\infty L(2\xi/Re_L)^{1/2}f(\xi,\eta)-\int_0^x \frac{r}{L}v_w\mathrm{d}x \tag{3}$$

where the stream function ψ satisfies the continuity equation, where $ru=L\partial\psi/\partial y$, $rv=-L\partial\psi/\partial x$, and Re_L is the Reynolds number $Re_L=u_\infty L/\nu$.

By the dimensionless variable transformation mentioned above, the transformed governing equations and their boundary conditions are

$$f'''+ff''+\Lambda(1-f'^2)+\frac{2\xi}{r}\frac{\mathrm{d}r}{\mathrm{d}\xi}\left(\frac{r^2\Omega^2}{Ue^2}\right)g^2$$

$$-\alpha(\xi)f''=2\xi\left(f'\frac{\partial f'}{\partial\xi}-f''\frac{\partial f}{\partial\xi}\right) \tag{4}$$

$$g''+fg'-gf'\left(\frac{4\xi}{r}\right)\frac{\mathrm{d}r}{\mathrm{d}\xi}-\alpha(\xi)g'$$

$$=2\xi\left(f'\frac{\partial g}{\partial\xi}-g'\frac{\partial f}{\partial\xi}\right) \tag{5}$$

$$Pr^{-1}\theta''+f\theta'-\alpha(\xi)\theta'$$

$$=2\xi\left(f'\frac{\partial\theta}{\partial\xi}-\theta'\frac{\partial f}{\partial\xi}\right) \tag{6}$$

with the boundary conditions

$$f=f'=0, \quad g=1, \quad \theta=1$$

$$\text{or} \quad \theta'=-\sqrt{2\xi}/(rUe/Lu_\infty) \quad \text{for } \eta=0 \tag{7a}$$

$$f'=1, \ \theta=0, \ g=0 \ \text{for } \eta\to\infty \tag{7b}$$

where

$$\alpha(\xi)=\frac{v_w}{Ue}\frac{L}{r}\sqrt{2\xi}Re_L^{1/2}$$

and

$$\Lambda=\frac{2\xi}{Ue}\frac{\mathrm{d}Ue}{\mathrm{d}\xi}$$